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On the calculation of rigidity characteristics of the stressed constructions

A.G. Kolpakov *

Bld. 95, 324, 9th November Str., Novosibirsk 630009, Russian Federation

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Abstract

It has been shown by Kolpakov (1989) that the homogenization method must be applied directly to the initial body in order to incorporate correctly the preliminary (initial) stresses if the body is non-homogeneous. In the paper mentioned above, this fact is noted and illustrated on examples. In the present paper, a complete analysis of the problem is given. In particular, the case of small initial stresses (as compared with elastic constants) is considered. This is the case realized in most natural and artificial stressed structures.

It was noted by Kolpakov (1992) that the realizing of the theoretical results represent an independent problem for every type of definite structure. In the present paper, the method of incorporating the initial stresses in application to finite-dimensional constructions (framework and semimonocoque constructions) is given.

Keywords: Stressed constructions; Rigidity; Homogenization

1. Homogenization for stressed non-homogeneous media

Consider a non-homogeneous elastic body having a periodic structure, which has periodicity cell (PC) P_ε (see Fig. 1). Here, $\varepsilon \ll 1$ denotes a characteristic dimension of the PC (see Fig. 1). The condition $\varepsilon \ll 1$ is formalized as $\varepsilon \to 0$.

The body is subjected to forces $\mathbf{F}$, which cause stresses $\sigma^\varepsilon_{ij}(0)$, which are called the initial ones. In applying the additional forces $\mathbf{f}$, the problem of deformation of a body having initial stresses appears.

The general description of a body with initial stresses has been considered in a book by Washizu (1982), and the following problems have been derived for describing the basic (initial) state:

$$L_\varepsilon(0)\mathbf{v}^\varepsilon = \mathbf{F} \text{ in } Q, \qquad \sigma^\varepsilon_{ij}(0)n^\varepsilon_j = 0 \text{ on } S_1 \cup S^\varepsilon, \qquad \mathbf{v}^\varepsilon = 0 \text{ on } S_2 \tag{1.1}$$

and for determining the additional displacements,

$$L_\varepsilon(\sigma)\mathbf{u}^\varepsilon = \mathbf{f} \text{ in } Q, \qquad \sigma^\varepsilon_{ij}(\sigma)n^\varepsilon_j = 0 \text{ on } S_1 \cup S^\varepsilon, \qquad \mathbf{u}^\varepsilon = 0 \text{ on } S_2. \tag{1.2}$$

* Tel.: +7-383-2-665280; fax: +7-383-2-661039.
E-mail address: agk@neic.nsk.su (A.G. Kolpakov).

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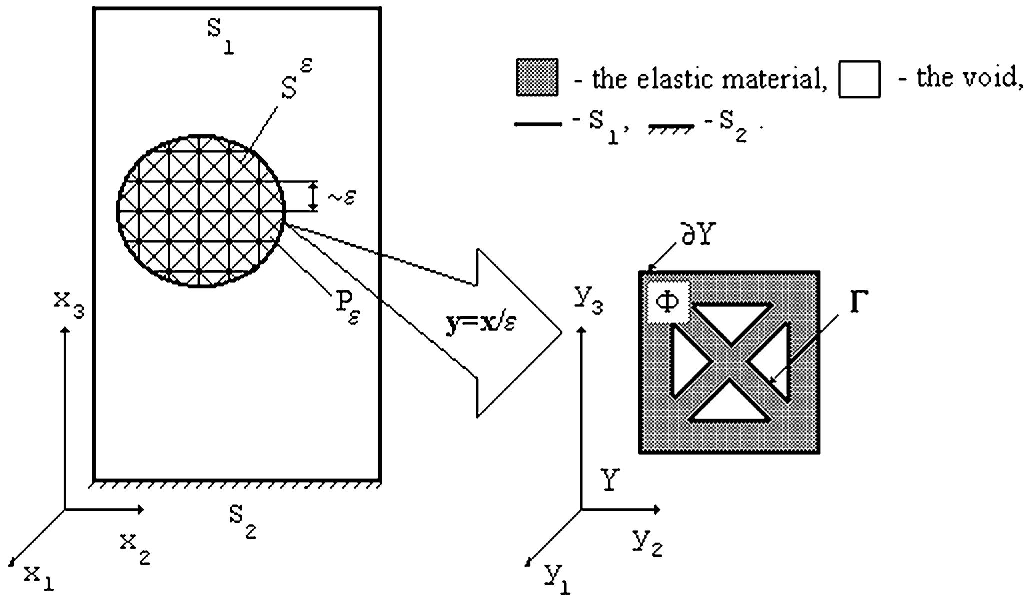

Fig. 1. Elastic body of periodic structure and the periodicity cell.

Here, $\mathbf{v}^\varepsilon$ and $\mathbf{u}^\varepsilon$ are the initial and additional displacements, respectively; $\delta_{ii} = 1$ and $\delta_{ik} = 0$ if $i \neq k$; $c_{ijkl}(\mathbf{x}/\varepsilon)$ are the local elastic constants;

$$\sigma^\varepsilon_{ij}(0) = c_{ijkl}(\mathbf{x}/\varepsilon)\,\partial v^\varepsilon_k/\partial x_1 \tag{1.3}$$

are the initial stresses;

$$\sigma^\varepsilon_{ij}(\sigma) = (c_{ijkl}(\mathbf{x}/\varepsilon) + \sigma^\varepsilon_{jl}(0)\delta_{ik})\,\partial u^\varepsilon_k/\partial x_1 \tag{1.4}$$

are the so-called additional stresses (Washizu, 1982); $L_\varepsilon(0)\mathbf{u} = \partial/\partial x_j[c_{ijkl}(\mathbf{x}/\varepsilon)\,\partial u_k/\partial xl]$ is the elasticity theory operator without the initial stresses; $L_\varepsilon(\sigma)\mathbf{u} = \partial/\partial x_j[(c_{ijkl}(\mathbf{x}/\varepsilon) + \sigma^\varepsilon_{jl}(0)\delta_{ik})\,\partial u_k/\partial xl]$ is the elasticity theory operator that incorporates the initial stresses; $\mathbf{n}^\varepsilon$ is the normal to $S_1 \cup S^\varepsilon$; S_1, S_2, S^ε are presented at Fig. 1, S^ε means the surface of voids (if exist).

The functions $c_{ijkl}(\mathbf{x}/\varepsilon)$, $\sigma^\varepsilon_{jl}(0)(\mathbf{x}, \mathbf{x}/\varepsilon)$ are periodic in $\mathbf{x}$ with PC P_ε according to the period of construction.

Summation with respect to the repeating indices is assumed.

Note. The elastic constants c_{ijkl} have well-known symmetries (in particular $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{klij}$, see Timoshenko and Goodier (1970)). Introducing $B_{ijkl} = c_{ijkl} + \sigma^\varepsilon_{jl}(0)\delta_{ik}$, we can write (1.4) in the form $\sigma^\varepsilon_{ij}(\sigma) = B_{ijkl}\partial u^\varepsilon_k/\partial x_1 l$ and consider it as a constitutive equation for a stressed body. In contrast to the elastic constants the quantities B_{ijkl} do not have symmetries occurring to the elastic constants. This asymmetry plays important role in the analysis of stressed inhomogeneous structures (Kolpakov, 1998, 2000).

1.1. Homogenization method as applied to stressed composites

It is known (Bensoussan et al., 1978) that the body as $\varepsilon \to 0$ can be replaced by a homogeneous (referred to as "homogenized") body similar to it in mechanical behavior. Correspondingly, the solution of Eqs. (1.1) and (1.2) may be approximated by the solution of the so-called homogenized problems:

$$L(0)\mathbf{v} = \mu\mathbf{F} \ \text{in} \ Q, \qquad \sigma_{ij}(0)n_j = 0 \ \text{on} \ S_1, \qquad \mathbf{v} = 0 \ \text{on} \ S_2, \tag{1.5}$$

$$L(\sigma)\mathbf{u} = \mu\mathbf{f} \ \text{in} \ Q, \qquad \sigma_{ij}(\sigma)n_j = 0 \ \text{on} \ S_1, \qquad \mathbf{u} = 0 \ \text{on} \ S_2. \tag{1.6}$$

Here,

- $\mathbf{v}$ and $\mathbf{u}$ are the "homogenized" displacements (i.e., the displacements determined from the homogenized problems);
- $L(0)\mathbf{v} = \partial/\partial x_j[a_{ijkl}(0)\,\partial v_k/\partial x_1]$ is the homogenized operator corresponding to Eq. (1.5);
- $L(\sigma)\mathbf{u} = \partial/\partial x_j[a_{ijkl}(\sigma)\,\partial u_k/\partial x_1]$ is the homogenized operator corresponding to Eq. (1.6);
- $a_{ijkl}(0)$ are the coefficients of operator $L(0)$ (these are the "homogenized" elastic constants of the body with no initial stresses);
- $a_{ijkl}(\sigma)$ are the coefficients of operator $L(\sigma)$ (these are the "homogenized" constants of the body with initial stresses);

$$\sigma_{ij}(\sigma) = a_{ijkl}(\sigma)\,\partial u_k/\partial x_1;$$

$$\sigma_{ij}(0) = a_{ijkl}(0)\,\partial v_k/\partial x_1;$$

$\langle\cdot\rangle = (\text{mes } Y)^{-1}\int_\Phi \mathrm{d}\mathbf{y}$ is the average value over the PC $Y = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon\colon \mathbf{x} \in P_\varepsilon\}$ in the "fast" variables $\mathbf{y} = \mathbf{x}/\varepsilon$; $\Phi \subset Y$ is the subdomain of the periodicity cell Y occupied by the elastic material, Γ means the free surface (if exists) in the "fast" variables (Fig. 1); μ is the volume fraction of the material: $\mu = \text{mes}\,\Phi/\text{mes}\,Y$ ($\mu = 1$ and $\Phi = Y$ for a monolithic body, and $0 < \mu < 1$ for a porous one).

If $\mathbf{f} = \rho(\mathbf{x}/\varepsilon)\,\partial^2\mathbf{u}^\varepsilon/\partial t^2(\rho(\mathbf{x}/\varepsilon)$ means the local density), one obtains the dynamical problem for stressed structure. If $\mathbf{f} = \omega^\varepsilon\rho(\mathbf{x}/\varepsilon)\mathbf{u}^\varepsilon$, one obtains the problem of free vibration of stressed structure. Here, ρ^ε is the local density, respectively, in Eq. (1.6) $\mathbf{F} = 0$, $\mathbf{F} = \langle\rho\rangle\,\partial^2\mathbf{u}/\partial t^2$, and $\mathbf{F} = \omega^2\langle\rho\rangle\mathbf{u}$.

Numerous authors (Bakhvalov and Panasenko, 1989; Bensoussan et al., 1978; Kalamkarov and Kolpakov, 1997; Oleinik et al., 1990; Sanchez-Palencia, 1980 and references in these books) presented the homogenization procedures for an elastic body with no initial stresses. The homogenization procedures for the porous body were presented by Oleinik et al. (1990), Cioranescu and Saint Jean Paulin (1979), Lions (1980).

In particular from the work of Oleinik et al. (1990), it is known that $\sigma_{ij}(0)$ are equal to average value of the initial stresses:

$$\sigma_{ij}(0) = \langle\sigma^\varepsilon_{ij}(0)\rangle \tag{1.7}$$

1.2. Computation of homogenized constants of a stressed body

To derive formulas for computing the homogenized constants of stressed body, we use the two-scale asymptotic expansion method (Bakhvalov and Panasenko, 1989). We use the following expansions:

Expansion for displacements

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + \cdots = \mathbf{u}^{(0)}(\mathbf{x}) + \sum_{k=1}^{\infty}\varepsilon^k\mathbf{u}^{(k)}(\mathbf{x},\mathbf{y}), \tag{1.8}$$

Expansion for stresses

$$\sigma^\varepsilon_{ij}(\sigma) = \sum_{k=0}^{\infty}\varepsilon^k\sigma^{(k)}_{ij}(\mathbf{x},\mathbf{y}). \tag{1.9}$$

Here, $\mathbf{x}$ are the "slow" variables, and $\mathbf{y} = \mathbf{x}/\varepsilon$ are the "fast" variables. The functions in the right-hand side of Eqs. (1.8) and (1.9) are assumed to be periodic in $\mathbf{y}$ with periodicity cell Y. Note that the term $\mathbf{u}^{(0)}(\mathbf{x})$ in Eq. (1.8) depends on the "slow" variable $\mathbf{x}$ only.

With the use of two-scale expansion, the differential operators are presented in the form of sum of operators in $\mathbf{x}$ and $\mathbf{y}$. For the function $Z(\mathbf{x},\mathbf{y})$ of the arguments $\mathbf{x}$ and $\mathbf{y}$, as in the right-hand sides of Eqs. (1.8) and (1.9), this representation takes the form:

$$\partial Z/\partial x_i = Z_{,ix} + \varepsilon^{-1} Z_{,iy}. \tag{1.10}$$

Here and below $,ix$ means $\partial/\partial x_i$ and $,iy$ means $\partial/\partial y_i$.

Substituting Eqs. (1.8) and (1.9) into Eq. (1.4), we obtain with allowance for Eq. (1.10)

$$\sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij}^{(k)} = \sum_{k=0}^{\infty} \varepsilon^k B_{ijmn}\left(u_{m,nx}^{(k)} + \varepsilon^{-1} u_{m,ny}^{(k)}\right), \quad k = 0, 1, \ldots, \tag{1.11}$$

where

$$B_{ijmn} = c_{ijmn} + \sigma_{jn}^{\varepsilon}(0)\delta_{im}. \tag{1.12}$$

Equating the terms with identical power of ε in Eq. (1.11), we obtain

$$\sigma_{ij}^{(k)} = B_{ijmn} u_{m,nx}^{(k)} + B_{ijmn} u_{m,ny}^{(k+1)}, \quad \mathrm{k} = 0, 1, \ldots \tag{1.13}$$

The equilibrium equation (Eqs. (1.2) and (1.4)) may be written in the term of stresses:

$$\partial\sigma_{ij}^{\varepsilon}(\sigma)/\partial x_j = f_j \text{ in } Q, \qquad \sigma_{ij}^{\varepsilon}(\sigma) n_j = 0 \text{ on } \qquad S_1 \cup S^{\varepsilon}. \tag{1.14}$$

Substituting (1.9) into the equilibrium equations (1.14), we obtain with allowance for (1.10)

$$\sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij,jx}^{(k)} + \sum_{k=0}^{\infty} \varepsilon^{k-1} \sigma_{ij,jy}^{(k)} = f_i \text{ in } Q, \qquad \sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij}^{(k)} n_j = 0 \text{ on } S_1 \cup S^{\varepsilon}. \tag{1.15}$$

Equating the terms with identical power of ε in Eq. (1.15), we obtain an infinite sequence of equations:

$$\sigma_{ij,jx}^{(0)} + \sigma_{ij,jy}^{(1)} = f_i \text{ and } \sigma_{ij,jx}^{(k)} + \sigma_{ij,jy}^{(k+1)} = 0 \text{ for } k > 0, \qquad \sigma_{ij,jy}^{(0)} = 0 \text{ in } Y, \qquad \sigma_{ij}^{(k)} n_j = 0 \text{ on } \Gamma \quad k = 0, 1, \ldots \tag{1.16}$$

Averaging Eq. (1.16) over the periodicity cell Y, we obtain an infinite sequence of the homogenized equilibrium equations, the first of which is the following:

$$\left\langle \sigma_{ij}^{(0)} \right\rangle_{,jx} = \mu f_i. \tag{1.17}$$

Here, we use equality $\langle \sigma_{ij,jy}^{(1)} \rangle = 0$, which follows from the well-known formula:

$$\int_Y \sigma_{ij,jy}^{(1)} \, \mathrm{d}\mathbf{y} = \int_{\partial Y} \sigma_{ij}^{(1)} n_j \, \mathrm{d}\mathbf{y} + \int_{\Gamma} \sigma_{ij}^{(1)} n_j \, \mathrm{d}\mathbf{y}.$$

The first integral is equal to zero by virtue of periodicity $\sigma_{ij}^{(1)}$ and anti-periodicity vector-normal $\mathbf{n}$. The second integral is equal to zero by virtue of condition $\sigma_{ij}^{(1)} n_{ij} = 0$ on Γ.

Let us consider the problem (1.13, $k = 0$), (1.16, $k = 0$) which can be written as

$$\left(B_{ijmn}(\mathbf{y}) u_{m,ny}^{(1)} + B_{ijkl}(\mathbf{y}) u_{m,nx}^{(0)}\right)_{,jy} = 0 \text{ in } Y, \tag{1.18}$$

$$\left(B_{ijmn}(\mathbf{y}) u_{m,ny}^{(1)} + B_{ijkl}(\mathbf{y}) u_{m,nx}^{(0)}\right) n_j = 0 \text{ on } \Gamma. \tag{1.19}$$

Allowing for the fact that the function of the argument $\mathbf{x}$ plays the role of a parameter in the problems in the variables $\mathbf{y}$ and $\mathbf{u}^{(0)}$ depends on $\mathbf{x}$, only, solution of the problem (1.18) and (1.19) with periodicity conditions can be found in the form:

$$\mathbf{u}^{(1)} = \boldsymbol{N}^{mn}(\mathbf{y})u^{(0)}_{m,nx}(x) + \boldsymbol{V}(\mathbf{x}). \tag{1.20}$$

Here, $V(\mathbf{x})$ is an arbitrary function of the argument $\mathbf{x}$, which does not influence the final equations, and the periodic function $N^{kl}(\mathbf{y})$ represents a solution of the following cellular problem:

$$\begin{aligned} &\left(B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y})\right)_{,j} = 0 \ \text{ in } Y, \\ &\left(B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y})\right)n_j = 0 \ \text{ on } \Gamma. \end{aligned} \tag{1.21}$$

$\boldsymbol{N}^{kl}(\mathbf{y})$ is periodic in $\mathbf{y}$ with the periodicity cell Y.

Substituting Eq. (1.20) into Eq. (1.13), we have

$$\sigma^{(0)}_{ij} = \left(B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y})\right)u^{(0)}_{k,lx}(\mathbf{x}). \tag{1.22}$$

Averaging Eq. (1.22) over the cell Y, we obtain the following homogenized constitutive equation:

$$\left\langle \sigma^{(0)}_{ij} \right\rangle = a_{ijkl}(\sigma)u^{(0)}_{k,lx}(\mathbf{x}), \tag{1.23}$$

where

$$a_{ijkl}(\sigma) = \left\langle B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y}) \right\rangle \tag{1.24}$$

are called the homogenized characteristics of the stressed body.

The homogenized equilibrium equation (1.17), the homogenized constitutive equation (1.23) and the boundary conditions,

$$\mathbf{u}^{(0)}(\mathbf{x}) = 0 \ \text{ on } S_2, \qquad \sigma^{(0)}_{ij}n^{\varepsilon}_j = 0 \ \text{ on } S_1, \tag{1.25}$$

represent the homogenized problem for stressed body. Substituting Eq. (1.24) into Eq. (1.23), we can write the homogenized problem in the form (1.6).

The fundamental difference of this problem from the homogenized problem for body having no initial stresses is the dependence of the cellular problem (1.21) and the homogenized coefficients $a_{ijkl}(\sigma)$ of the initial stresses.

Note, that in general case (Kolpakov, 1989, 1992)

$$a_{ijkl}(\sigma) \neq a_{ijkl}(0) + \sigma_{jl}(0)\delta_{ik}. \tag{1.26}$$

The right-hand side in Eq. (1.26) arises when one uses the so-called "intermediate" homogenization, which is carried out as follows: one homogenizes the non-homogeneous body having no initial stresses and calculates the stresses in it by solving the problem (1.5), and then one compiles an operator that should arise in describing a real homogeneous body having those elastic constants and initial stresses in accordance with the classical theory presented, e.g., in the book by Washizu (1982). The intermediate homogenization arises in particular from a phenomenological approach to a non-homogeneous body. In this case, the experimentally measured elastic constants are the homogenized ones. It follows from Eq. (1.26) that intermediate homogenization in general leads to an incorrect result. Mathematically, this is due to the fact that the G limit of a sum is not equal to the sum of G limits (Marcellini, 1975). From the mechanical viewpoint, it is explained by the occurrence of a general state of local stress and strain when the uniform homogenized stresses are applied to a non-homogeneous medium.

2. Small initial stresses

These are naturally constraints on the initial stresses, i.e., $\sigma^{\varepsilon}_{ij}(0)$ will not exceed the strength limit of the material. In turn, the strength limit for a real material is small by comparison with the elastic constants (Timoshenko, 1955). Then, the initial stresses $\sigma^{\varepsilon}_{ij}(0)$ are small compared with elastic constants c_{ijkl} and values B_{ijkl} introduced by formula (1.12) may be presented as

$$B_{ijkl}(\mathbf{x},\mathbf{y}) = c_{ijkl}(\mathbf{y}) + \mu b_{ijkl}(\mathbf{x},\mathbf{y}), \tag{2.1}$$

where μ is a small parameter ($\mu \sim 0.01$) and the following notation is used: $b_{ijkl}(\mathbf{x},\mathbf{y}) = \sigma^{\varepsilon}_{jl}(0)(\mathbf{x},\mathbf{y})\delta_{ik}$. In order to solve the cellular problem (1.21) with coefficients (2.1), classical method of small parameter can be used. This method is based on presentation of solution of the cellular problem in the form:

$$\boldsymbol{N}^{kl}(\mathbf{y}) = \boldsymbol{N}^{0kl}(\mathbf{y}) + \mu\boldsymbol{N}^{lkl}(\mathbf{y}) + \cdots = \sum_{s=0}^{\infty}\mu^{s}\boldsymbol{N}^{skl}(\mathbf{y}). \tag{2.2}$$

All the functions $N^{skl}(\mathbf{y})$ are assumed to be periodic in $\mathbf{y}$ with the periodicity cell Y.

Substituting Eq. (2.2) into Eq. (1.21) and equating the terms with identical power of μ, we obtain an infinite sequence of problems, the first two of which have the following form:

$$\Big(c_{ijnm}(\boldsymbol{y})N^{0kl}_{m,ny} + c_{ijkl}(\boldsymbol{y})\Big)_{,jy} = 0 \;\text{ in } Y, \tag{2.3}$$

$$\Big(c_{ijnm}(\boldsymbol{y})N^{0kl}_{m,ny} + c_{ijkl}(\boldsymbol{y})\Big)n_j = 0 \;\text{ on } \Gamma, \tag{2.4}$$

$$\Big(c_{ijnm}(\boldsymbol{y})N^{lkl}_{m,ny} + b_{ijkl}(\boldsymbol{x},\boldsymbol{y}) + b_{ijkl}(\boldsymbol{x},\boldsymbol{y})N^{0kl}_{m,ny}(\boldsymbol{y})\Big)_{,jy} = 0 \;\text{ in } Y, \tag{2.5}$$

$$\Big(c_{ijnm}(\boldsymbol{y})N^{1kl}_{m,ny} + b_{ijkl}(\boldsymbol{x},\boldsymbol{y}) + b_{ijkl}(\boldsymbol{x},\boldsymbol{y})N^{0kl}_{m,ny}(\boldsymbol{y})\Big)n_j = 0 \;\text{ on } \Gamma. \tag{2.6}$$

Problem (2.3) and (2.4) is the well-known cellular problem for a body with no initial stresses (Bensoussan et al., 1978; Kalamkarov and Kolpakov, 1997; Oleinik et al., 1990; Sanchez-Palencia, 1980). We can obtain the problem (2.3), (2.4) from the cellular problem (1.21), if we put $B_{ijkl} = c_{ijkl}$ in (1.21).

Transform formula (1.24) to a quadratic functional. For that we change in Eq. (1.21) indices ($ij \longleftrightarrow pq$), then multiply the Eq. (1.21) by N^{ij}_{p} and integrate by parts over the periodicity cell Y. As a result, we obtain with allowance for periodicity of $\boldsymbol{N}^{kl}$ and $\boldsymbol{N}^{pq}$ the following equality:

$$\Big\langle B_{pqnm}(\mathbf{y})N^{kl}_{m,ny}N^{ij}_{p,qy} + B_{pqkl}(\mathbf{x},\mathbf{y})N^{ij}_{p,qy}\Big\rangle = 0. \tag{2.7}$$

Subtracting Eq. (2.7) from Eq. (1.24), we obtain with regard for definition B_{ijkl} (2.1) and symmetry $c_{ijkl} = c_{klij}$ the following formula:

$$\begin{aligned} a_{ijkl}(\sigma) &= \Big\langle -B_{pqmn}(\mathbf{y})N^{ij}_{m,ny}N^{kl}_{p,qy} - B_{mnij}(\mathbf{x},\mathbf{y})N^{kl}_{m,ny} + B_{ijmn}(\mathbf{x},\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{x},\mathbf{y})\Big\rangle \\ &= \Big\langle -B_{pqnm}(\mathbf{y})N^{ij}_{m,ny}N^{kl}_{p,qy} - \mu b_{mnij}(\mathbf{x},\mathbf{y})N^{kl}_{m,ny} + \mu \mathrm{b}_{ijmn}(\mathbf{x},\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{x},\mathbf{y})\Big\rangle. \end{aligned} \tag{2.8}$$

Proposition 1. *Let stresses σ^{*}_{ij} are periodic in* $\mathbf{y}$ *with the periodicity cell Y and satisfy the following equations: $\sigma^{*}_{ij,jy} = 0$ in Y, $\sigma^{*}_{ij}n_j = 0$ on Γ. Then, $\langle\sigma^{*}_{ij}Z_{,jy}\rangle = 0$ for any function $Z(\mathbf{y})$ periodic in* $\mathbf{y}$ *with the periodicity cell Y.*

To prove the Proposition 1, let us multiply the first equation by $Z(\mathbf{y})$ and integrate the result by parts over the cell Y. We have

$$0 = \int_Y \sigma^*_{ij} Z_{,jy} \mathrm{d}\mathbf{y} + \int_{\partial Y} \sigma^*_{ij} Z n_j \, \mathrm{d}\mathbf{y} + \int_\Gamma \sigma^*_{ij} Z n_j \, \mathrm{d}\mathbf{y}.$$

The second integral is equal to zero by virtue of periodicity $\sigma^*_{ij}(\mathbf{y})$ and $Z(\mathbf{y})$ and anti-periodicity of the vector-normal n. The third integral is equal to zero by virtue of the boundary condition.

Proposition 2. *The initial stresses $\sigma^\varepsilon_{ij}(0)$ determined from solution of the elasticity problem* (1.1) *and* (1.2) *satisfy the conditions of the Proposition 1.*

To prove the proposition, we use the following well-known (Bakhvalov and Panasenko, 1989) representation for the stresses:

$$\sigma^\varepsilon_{ij}(0) = c_{ijkl}(\mathbf{y}) + \left(v_{k,lx}(\mathbf{x}) + N^{0kl}_{m,ny}(\mathbf{y}) v_{k,lx}(\mathbf{x}) \right), \tag{2.9}$$

where $\mathbf{v}$ is the solution of the homogenized problem (1.5).

Using Eq. (2.9), we obtain $\sigma^\varepsilon_{ij,jy}(0) = (c_{ijkl}(\mathbf{y}) + c_{ijmn}(\mathbf{y}) N^{0kl}_{m,ny}(\mathbf{y}))_{,jy} \, v_{k,ly}(\mathbf{x})$. The right-hand side of this equality is equal to zero (it is the left-hand side of the cellular equation for the body having no initial stresses, see Eq. (1.21)).

From Eq. (2.9), we also have $\sigma^\varepsilon_{ij}(0) n_j = (c_{ijkl}(\mathbf{y}) + c_{ijmn}(\mathbf{y}) N^{kl}_{m,ny}(\mathbf{y}) v_{m,nx}(\mathbf{x})) n_j v_{k,lx}(\mathbf{x})$. The right-hand side of this equality is equal to zero (it is the left-hand side of the cellular boundary condition for the body having no initial stresses, see Eq. (1.21)). The periodicity of $\sigma^\varepsilon_{ij}(0)$ in $\mathbf{y}$ follows from periodicity of $c_{ijmn}(\mathbf{y})$ and $\boldsymbol{N}^{kl}(\mathbf{y})$.

By virtue of Proposition 1, definition of b_{ijkl} and symmetry of the initial stresses $\sigma^\varepsilon_{ij}(0)$ with respect to indices i and j, we obtain

$$\begin{aligned} \left\langle b_{mnij} N^{kl}_{m,ny} \right\rangle &= \left\langle \sigma^\varepsilon_{nj}(0) N^{kl}_{m,ny} \right\rangle \delta_{im} = \left\langle \sigma^\varepsilon_{jn}(0) N^{kl}_{m,ny} \right\rangle \delta_{im} = 0, \\ \left\langle b_{ijmn} N^{kl}_{m,ny} \right\rangle &= \left\langle \sigma^\varepsilon_{jn}(0) N^{kl}_{m,ny} \right\rangle \delta_{im} = 0. \end{aligned} \tag{2.10}$$

From Eqs. (2.8) and (2.10), we obtain

$$a_{ijkl}(\sigma) = \left\langle - B_{pqmn}(\mathbf{y}) N^{ij}_{m,ny} N^{kl}_{p,qy} + B_{ijkl} \right\rangle. \tag{2.11}$$

Substituting decomposition (2.1) into formula (2.11) and saving only terms linear in μ, we obtain

$$\begin{aligned} a_{ijkl}(\sigma) &= a_{ijkl}(0) + \mu[\langle b_{ijkl}(\mathbf{x},\mathbf{y})\rangle + l_{ijkl}(\sigma)] + \cdots = a_{ijkl}(0) + \mu\left[\left\langle \sigma^\varepsilon_{jl}(0)(\mathbf{x},\mathbf{y}) \right\rangle \delta_{ik} + l_{ijkl}(\sigma)\right] + \cdots \\ &= a_{ijkl}(0) + \mu[\sigma_{jl}(0)(\mathbf{x},\mathbf{y}) \delta_{ik} + l_{ijkl}(\sigma)] + \cdots, \end{aligned} \tag{2.12}$$

where

$$l_{ijkl}(\sigma) = \left\langle - b_{pqmn} N^{0ij}_{p,qy} N^{0kl}_{m,ny} - c_{pqmn} N^{0ij}_{p,qy} N^{1kl}_{m,ny} - c_{pqmn} N^{1ij}_{m,ny} N^{0kl}_{p,qy} \right\rangle. \tag{2.13}$$

Writing the last equality in Eq. (2.12), we use the equality (1.7). By resorting to problem (2.5) and (2.6), we can take the opportunity to rule out functions N^{1kl} from Eq. (2.13).

Multiplying Eq. (2.5) by N_i^{0pq} and integrating the result by parts over the cell Y, we obtain with allowance for periodicity of $\boldsymbol{N}^{0ij}$ and $\boldsymbol{N}^{1kl}$ and the boundary condition (2.6)

$$\left\langle c_{ijnm}(\mathbf{y})N^{lkl}_{m,ny}N^{0pq}_{i,jy} + b_{ijkl}(\mathbf{x},\mathbf{y})N^{0pq}_{i,jy} + b_{ijkl}(\mathbf{x},\mathbf{y})N^{0kl}_{m,ny}N^{0pq}_{i,jy}\right\rangle = 0. \tag{2.14}$$

From Eq. (2.14), we have (after changing the indices $ij \longleftrightarrow pq$)

$$\left\langle c_{pqmn}(\mathbf{y})N^{lkl}_{m,ny}N^{0ij}_{p,qy}\right\rangle = -\left\langle b_{pqkl}(\mathbf{x},\mathbf{y})N^{0ij}_{p,qy} + b_{pqmn}(\mathbf{x},\mathbf{y})N^{0kl}_{m,ny}N^{0ij}_{p,qy}\right\rangle. \tag{2.15}$$

In the same way, we have

$$\left\langle c_{pqnm}(\mathbf{y})N^{0kl}_{m,ny}N^{lij}_{p,qy}\right\rangle = -\left\langle b_{pqij}(\mathbf{x},\mathbf{y})N^{0kl}_{p,qy} + b_{pqmn}(\mathbf{x},\mathbf{y})N^{0kl}_{m,ny}N^{0ij}_{p,qy}\right\rangle. \tag{2.16}$$

Substituting Eqs. (2.15) and (2.16) into Eq. (2.13), we obtain

$$l_{ijkl}(\sigma) = \left\langle b_{pqmn}N^{0kl}_{m,ny}N^{0ij}_{p,qy} + b_{pqkl}N^{0ij}_{p,qy} + b_{pqij}N^{0ij}_{p,qy}\right\rangle. \tag{2.17}$$

The following equalities takes place:

$$\left\langle b_{pqkl}N^{0ij}_{p,qy}\right\rangle = 0 \text{ and } \left\langle b_{pqij}N^{0ij}_{p,qy}\right\rangle = 0. \tag{2.18}$$

The equalities (2.18) can be derived in a manner similar to one used to derive equalities (2.10). From Eqs. (2.17) and (2.18), we obtain

$$l_{ijkl}(\sigma) = \left\langle b_{pqmn}N^{0kl}_{m,ny}N^{0ij}_{p,qy}\right\rangle. \tag{2.19}$$

Substituting $b_{ijmn} = \sigma^{\varepsilon}_{jn}(0)\delta_{im}$ in accordance with the definition (2.1), we obtain

$$l_{ijkl}(\sigma) = \left\langle \sigma^{\varepsilon}_{qn}(0)N^{0kl}_{p,ny}N^{0ij}_{p,qy}\right\rangle. \tag{2.20}$$

Note. The l_{ijkl} (σ) are expressed in terms of derivatives of $N^{0\alpha\beta}$ and cannot be expressed in terms of deformations corresponding to $N^{0\alpha\beta}$ in the general case.

The formula (2.20) can be written in terms of the homogenized stresses. In accordance with Oleinik et al. (1990), the local stresses in a body with no initial stresses are given by formula:

$$\sigma^{\varepsilon}_{ij}(0) = c_{ijkl}(\mathbf{x}/\varepsilon)\left(e_{kl} + N^{0pq}_{k,ly}(\mathbf{x}/\varepsilon)e_{pq}\right) = c_{ijkl}(\mathbf{x}/\varepsilon)\left(\delta_{kp}\delta_{lq} + N^{0pq}_{k,ly}(\mathbf{x}/\varepsilon)\right)\boldsymbol{J}_{pqmn}\sigma_{mn}(0), \tag{2.21}$$

where $e_{pq} = 1/2(\partial v_p/\partial x_q + \partial v_q/\partial x_p)$ are the homogenized strains and $\{\boldsymbol{J}_{pqmn}\} = \{a_{ijkl}(0)\}^{-1}$ is the homogenized compliance tensor.

Substituting the last expression from Eq. (2.21) into Eq. (2.20) in place of $\sigma^{\varepsilon}_{ij}$, we obtain the following formula:

$$a_{ijkl}(\sigma) = a_{ijkl}(0) + \mu[\sigma_{jl}(0)\delta_{ik} + l_{ijklrs}(\sigma)\boldsymbol{J}_{rsmn}\sigma_{mn}(0)] \tag{2.22}$$

in which

$$l_{ijklrs}(\sigma) = \left\langle c_{qtcd}N^{0rs}_{c,dy}N^{0kl}_{p,ty}N^{0ij}_{p,qy} + c_{qtrs}N^{0kl}_{p,ty}N^{0ij}_{p,qy}\right\rangle \tag{2.23}$$

with summation with respect to the repeating subscripts.

2.1. Elastic constants of a stressed body

It follows from Eqs. (2.3) and (2.4) and the elastic-constants $c_{ijkl}(\mathbf{y})$ symmetry that $\boldsymbol{N}^{ij}$ is symmetrical with respect to the superscripts. Then, the quantities $l_{ijkl}(\sigma)$ have the symmetries occurring in the elastic constants. It is known (Bensoussan et al., 1978; Sanchez-Palencia, 1980; Kalamkarov and Kolpakov, 1997) that the homogenized coefficients a_{ijkl} (0) have the symmetries occurring in the elastic constants. Then, the quantities

$$A_{ijkl}(\sigma) = a_{ijkl}(0) + \mu l_{ijkl}(\sigma) \tag{2.24}$$

have the symmetries occurring in the elastic constants and can be interpreted as the homogenized elastic constants of stressed body.

In terms of the quantities $A_{ijkl}(\sigma)$, the formula (2.12) can be written in the form:

$$a_{ijkl}(\sigma) = A_{ijkl}(\sigma) + \mu\sigma_{jl}(0)\delta_{ik} \tag{2.25}$$

coinciding in form with the classical formula for homogeneous body having initial stresses (see e.g. Washizu (1982) and compare with formula for $L_\varepsilon(\sigma)$ above). Note that the so-introduced elastic constants $A_{ijkl}(\sigma)$ of homogenized body depend on the initial stresses.

The initial stresses can exist both in homogeneous and non-homogeneous bodies. For homogeneous body $c_{ijkl} = \text{const}$, and solution of cellular problem (2.3) and (2.4) $\boldsymbol{N}^{\alpha\beta} = 0$. Then, $l_{ijkl}(\sigma) = 0$ in accordance with Eq. (2.20). It is possible that $\boldsymbol{N}^{\alpha\beta} = 0$ and then $l_{ijkl}(\sigma) = 0$ for a non-homogeneous body. It takes place in the case when the local displacements coincide with the global displacements. Consequently, the effect discovered arises in non-homogeneous bodies only (but not in any non-homogeneous body).

It is possible that the local stresses $\sigma^\varepsilon_{ij}(0) \neq 0$ although the homogenized (averaged) stresses $\sigma_{ij}(0) = \langle\sigma^\varepsilon_{ij}(0)\rangle = 0$. In this case, $\sigma^\varepsilon_{ij}(0)$ are called "self-balanced" stresses. In this case, the homogenized problem coincides with the elasticity problem for a body having no initial stresses and having elasticity constants $A_{ijkl}(\sigma)$ given by formula (2.24) (see Eqs. (2.25) and (1.6) and formula for $L(\sigma)$ above). Consequently, the self-balanced stresses can affect the homogenized characteristics of non-homogeneous body and cannot affect the characteristics of homogeneous body.

The constructions considered in the following sections of the paper are the non-homogeneous bodies (one component is material, the other component is voids) even made of homogeneous material. Both cases $l_{ijkl}(\sigma)$ equalling zero and not equalling zero can be realized for these. The examples will be given.

3. The stressed structures made of rectilinear or planar elements

We consider a construction of periodic structure formed of beams, plates, and rods. Such a structure is a particular case of highly porous framework structure. In this case, Eqs. (2.3) and (2.4) can be replaced as proposed by Kolpakov (1985) by a cellular problem for the corresponding cellular constructions formed by a system of beams and/or plates (Annin et al., 1993; Kalamkarov and Kolpakov, 1996, 1997).

The problem (2.3) and (2.4) can be considered as a problem with respect to $\mathbf{U}^{\alpha\beta} = \boldsymbol{N}^{\alpha\beta} + y_\alpha\mathbf{e}_\beta$ (the cellular problem theory for beams/plates is formulated naturally in terms of $\mathbf{U}^{\alpha\beta}$ in the sense that the kinematic hypotheses link the displacements of the cellular construction elements to $\mathbf{U}^{\alpha\beta}$). One can rewrite Eqs. (2.20) and (2.22) in terms of $\mathbf{U}^{\alpha\beta}$:

$$l_{ijkl}(\sigma) = \left\langle \sigma^\varepsilon_{qm}(0)\left(U^{kl}_{p,my} - \delta_{kp}\delta_{lm}\right)\left(U^{ij}_{p,qy} - \delta_{kp}\delta_{lq}\right)\right\rangle, \tag{3.1}$$

and correspondingly

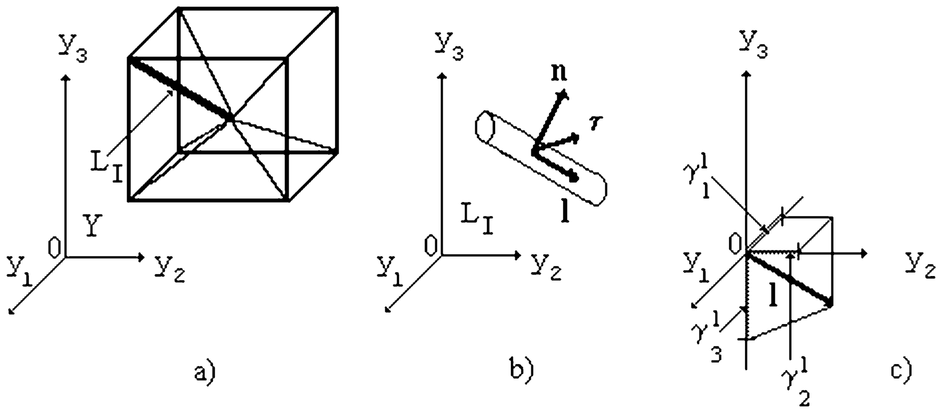

Fig. 2. (a) The periodicity cell Y formed by rods, (b) the beam L_I and the vectors $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{\tau}$ of local coordinate system linked to the beam, and (c) the directional vector $\boldsymbol{l}$ and the coordinate cosines (these are projections of the vector $\boldsymbol{l}$ on the coordinate axes.

$$l_{ijklrs}(\sigma) = \Big\langle c^{qtcd}\Big(U^{rs}_{c,dy} - \delta_{rc}\delta_{sd}\Big)\Big(U^{kl}_{p,ty} - \delta_{kp}\delta_{lt}\Big)\Big(U^{ij}_{p,qy} - \delta_{ip}\delta_{jm}\Big) + c_{qtrs}\Big(U^{kl}_{p,ty} - \delta_{kp}\delta_{lm}\Big)\Big(U^{ij}_{p,qy} - \delta_{ip}\delta_{jm}\Big)\Big\rangle. \tag{3.2}$$

The theory of beams/plates establishes a relation between the displacements of the elements (considered as 1-D or 2-D objects) and $\mathbf{U}^{\alpha\beta}$ (displacements of the elements considered as solid bodies) in the simplest form in the natural coordinate system linked to the elements (Timoshenko and Woinowsky-Krieger, 1959; Washizu, 1982; Haug et al., 1986). We introduce the $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{\tau}\}_I$ coordinate system linked to body I (Fig. 2). The vector $\boldsymbol{l}$ is the unit length direction vector for beams; $\boldsymbol{n}$, the unit length normal vector; $\boldsymbol{\tau}$, the unit length vector perpendicular to $\boldsymbol{l}$, $\boldsymbol{n}$. For the planar structures $\{\boldsymbol{l}, \boldsymbol{n}\}_I$ system is used.

Let us denote by $\{\gamma^i_a\}$, the directional cosines of $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{\tau}\}_I$ coordinate basis vectors relative to the coordinate system $Oy_1y_2y_3$, $i = 1, 2, 3$; $A = l, n, \tau$ (Fig. 2). Applying the standard formulas for tensors transformations when the coordinate system is changed (Washizu, 1982), one can rewrite formulas (3.1) and (3.2) in a coordinate system linked to element as follows:

$$l_{ijkl}(\sigma) = \Big\langle \sigma^{\varepsilon}_{QM}(0)U^{kl}_{P,My}U^{ij}_{P,Qy} - \gamma^k_A\gamma^l_B\sigma^{\varepsilon}_{QB}(0)U^{ij}_{A,Qy} - \gamma^i_A\gamma^j_B\sigma^{\varepsilon}_{BM}U^{kl}_{A,My} + \gamma^i_A\gamma^j_B\gamma^r_R\gamma^s_S\sigma^{\varepsilon}_{BS}(0)\delta_{AR}\Big\rangle \tag{3.3}$$

and

$$l_{ijklab}(\sigma) = \Big\langle c_{QLCD}\Big(U^{ab}_{C,Dy} - \delta_{aC}\delta_{bD}\Big)\Big(U^{kl}_{P,Sy} - \delta_{kP}\delta_{lS}\Big)\Big(U^{ij}_{P,Qy} - \delta_{iP}\delta_{jQ}\Big) + \gamma^a_R\gamma^b_S c_{QLRS}\Big(U^{kl}_{P,Sy} - \delta_{kP}\delta_{lS}\Big)\Big(U^{ij}_{P,Qy} - \delta_{iP}\delta_{jQ}\Big)\Big\rangle. \tag{3.4}$$

Here, the capital letter indices run through the values l, n, τ and the small letter indices through values 1, 2, 3.

The average value over the PC Y in the present case is

$$\langle\cdot\rangle = (\operatorname{mes} Y)^{-1}\sum_{I=1}^{N}\int_{L_I} \mathrm{d}\mathbf{y},$$

where N is the number of elements (beams and/or plates) in the cellular construction; the integration is taken over the region L_I occupied by the Ith element I. The integrals can be calculated explicitly on the basis of the hypotheses from beam/plate theory.

In all the calculations below, structural elements made of isotropic materials will be considered. As a result the stress tensor and strain tensor are co-axial.

3.1. Periodic beam structures

Consider a construction made of thin beams. We write the following formulas for the stresses in the coordinate system linked to beam I (the beam material is taken as homogeneous and isotropic, and the beam has a constant cross-section):

$$\sigma^{\varepsilon}_{ll}(0) \neq 0, \quad \sigma^{\varepsilon}_{AB}(0) = 0 \text{ for } AB \neq ll. \tag{3.5}$$

We represent Eqs. (3.3) and (3.4) on the basis of Eq. (3.5) as

$$l_{ijrs}(\sigma) = (\text{mes}\, Y)^{-1}\sum_{I=1}^{N} \int_{L_I} \sigma^{\varepsilon}_{ll}(0)\left[U^{rs}_{A,l}U^{ij}_{A,l} - \gamma^r A\gamma^s_l U^{ij}_{A,l} - \gamma^i_A\gamma^j_l U^{rs}_{A,l} + \gamma^i_l\gamma^j_l\gamma^r_l\gamma^s_l\right] \mathrm{d}l\mathrm{d}n\,\mathrm{d}\tau. \tag{3.6}$$

For the case under consideration $\sigma^{\varepsilon}_{ll}(0) = E_I U^{ab}_{l,l} v_{a,b}$ (compare with Eq. (2.21)). E_I is Young's modulus of the material of the Ith beam. Then

$$l_{ijrsab}(\sigma) = (\text{mes}\, Y)^{-1}\sum_{I=1}^{N} \int_{L_I} E_I\left[U^{ab}_{l,l}U^{rs}_{A,l}U^{ij}_{A,l} - U^{ab}_{l,l}\left(\gamma^r_A\gamma^s_l U^{ij}_{A,l} + \gamma^i_A\gamma^j_l U^{rs}_{A,l}\right) + U^{ab}_{l,l}\gamma^i_l\gamma^j_l\gamma^r_l\gamma^s_l\right] \mathrm{d}l\mathrm{d}n\,\mathrm{d}\tau. \tag{3.7}$$

The stresses and strains in Eq. (3.5) are the sum of stretching–compression strains and bending strains and take the form $A + Bn + C\tau$, where $A,B,C =$ const. Then, one can calculate the expressions in Eqs. (3.6) and (3.7) by integrating functions of the form $n^K\tau^L$, K, $L =$ integers, over the cross-section of the beam. An example will be given below.

3.2. Periodic rod structures

Let the bending stresses and strains in the cellular construction be negligible. Then, the stresses $\sigma^{\varepsilon}_{ll}(0) = E_I e^{ab}_{ll}$ and axial strains $e^{rs}_I = U^{rs}_{l,l}$ in Eqs. (3.5)–(3.7) are constants, and $U^{rs}_{A,l} = 0$ for $A \neq l$, so Eqs. (3.6) and (3.7) after integration become

$$l_{ijrs}(\sigma) = (\text{mes}\, Y)^{-1}\sum_{I=1}^{N} N_I\left[e^{rs}_I e^{ij}_I - \gamma^r_l\gamma^s_l e^{ij}_I - \gamma^i_l\gamma^j_l e^{rs}_I + \gamma^i_l\gamma^j_l\gamma^r_l\gamma^s_l\right]L_I, \tag{3.8}$$

$$l_{ijrsab}(\sigma) = (\text{mes}\, Y)^{-1}\sum_{I=1}^{N} E_I C_I\left[e^{ab}_I e^{rs}_I e^{ij}_I + e^{ab}_I\left(\gamma^r_l\gamma^s_l e^{ij}_I + \gamma^i_l\gamma^j_l e^{rs}_I\right) + e^{ab}\gamma^i_l\gamma^j_l\gamma^r_l\gamma^s_l\right]L_I \tag{3.9}$$

in which N_I is the initial axial force (the initial axial stress $\sigma^{\varepsilon}_{ll}$ multiplied by the cross-section area C_I of the Ith rod), L_I the length, and E_I the tensional rigidity of the Ith rod (Young's modulus E_I multiplied by C_I).

3.3. Semimonocoque constructions

Consider a construction made of thin rods and plates. Let the rod framework in a cellular construction support all the tension/compression loads, while the plates support only shear loads (Washizu, 1982). The rod framework was considered in Section 3.1. Now consider the plate shell. For plate subjected to shearing only, one can introduce a coordinate system linked to the plate in which $\sigma^{\varepsilon}_{AB}(0) = \text{const}$ for $AB = ln, nl$, $\sigma^{\varepsilon}_{AB}(0) = 0$ for $AB \neq ln, nl$; $U^{ij}_{l} = \Gamma^{ij} n, U^{ij}_{n} = \Gamma^{ij} l (\Gamma^{ij}$ is the shear strain).

As a result, for the plates formula (3.3) takes the form

$$l_{ijrs}(\sigma) = (\text{mes}\, Y)^{-1} \sum_{I=1}^{M} S_I \{ -(\gamma^r_n \gamma^s_n \Gamma^{ij} + \gamma^r_l \gamma^s_l \Gamma^{ij} + \gamma^i_n \gamma^j_n \Gamma^{rs} + \gamma^r_l \gamma^s_l \Gamma^{rs}) + (\gamma^i_A \gamma^j_l \gamma^r_R \gamma^s_n + \gamma^i_A \gamma^j_l \gamma^r_R \gamma^s_l) \delta_{AR} \} L_I. \tag{3.10}$$

Here, S_I are the initial shearing forces (the initial shearing stresses $\sigma^{\varepsilon}_{ln}$ multiplied by the plate thickness) and $\mathscr{L}_I$ is the area of the Ith plate in plane. We use here that $U^{rs}_{P,l} U^{ij}_{P,n} = 0$ because $U^{rs}_{l,l} = U^{ij}_{l,l} = U^{rs}_{n,n} = U^{ij}_{n,n} = 0$ in the case under consideration. The summation in (3.10) is taken over the number M of plates in the cellular construction. One should add (3.10) to (3.7) or (3.8) to get the final expression for l_{ijrs}.

4. Method solving the cellular problem

The most general method of determining $\boldsymbol{N}^{\alpha\beta}$ and $\mathbf{U}^{\alpha\beta}$ is to solve Eqs. (2.3) and (2.4) numerically. However, if the cellular construction is formed by thin-walled elements, it is logical to use methods that explicitly incorporate the thinwallness of the cellular construction elements. The approach proposed by Kolpakov (1985) is one such method. The method proposed there involves replacing the cellular problem in elasticity theory by the cellular problem in the theory of beams/plates and agrees with the analysis method for finite-dimensional structures that has been thoroughly developed in the sense of theoretical analysis and in the sense of software. We consider applying the last method to the cellular problem. We introduce the generalized displacements of the cellular construction nodes $(\mathbf{u}_1, \mathbf{m}_1, \ldots)$, in which $\mathbf{u}_1, \ldots$ are the displacements proper and $\mathbf{m}_1, \ldots$ are the residual components of the generalized-displacement vector (e.g., the angles of rotation for the ends of the beams and so on (Washizu, 1982; Haug et al., 1986). The finite-dimensional cellular problem takes the form:

$$\mathbf{T}\mathbf{w}^{\alpha\beta} = 0 \quad \text{at the interior nodes}, \tag{4.1}$$

$$(\mathbf{T}\mathbf{w}^{\alpha\beta})_{a+} = (\mathbf{T}\mathbf{w}^{\alpha\beta})_{a-} \quad \text{at the boundary nodes}, \tag{4.2}$$

$$(\mathbf{w}^{\alpha\beta} - y_\alpha \mathbf{e}_\beta)_{a+} = (\mathbf{w}^{\alpha\beta} - y_\alpha \mathbf{e}_\beta)_{a-} \quad \text{at the boundary nodes}, \tag{4.3}$$

$$\sum_{I=1}^{N} (\mathbf{w}^{\alpha\beta} - y_I \mathbf{e}_\beta) = \sum_{I=1}^{N} \mathbf{m}^{\alpha\beta}_I = 0. \tag{4.4}$$

Here, Eq. (4.1) are the equations of equilibrium ($\mathbf{T}$ is the influence/stiffness matrix); Eqs. (4.2) and (4.3) are the periodicity conditions (the subscripts $a+$ and $a-$ denote those corresponding one another at opposite faces of the PC); Eq. (4.4) is the analog of $\langle \boldsymbol{N}^{\alpha\beta} \rangle = 0$; N is the number of cellular construction elements; y_α $\mathbf{e}_\beta$ takes the values at the nodes of the PC.

One solves Eqs. (4.1)–(4.4), then recovers $\mathbf{w}^{\alpha\beta}$ in the region occupied by the elements on the basis of the kinematic hypothesis, and calculates l_{ijrs} or l_{ijrsab} in accordance with the above formulas. For typical constructional elements, such as rods, beams and plates, one can obtain explicit expressions for l_{ijrs} and l_{ijrsmn} in terms of the generalized displacements of the ends.

To calculate $a_{ij\alpha\beta}(0)$, one can apply the following formula for computing the homogenized elastic constants presented in the book by Kalamkarov and Kolpakov (1997):

$$a_{ij\alpha\beta}(0) = \sum_{Gj} (\mathbf{T}\mathbf{w}^{\alpha\beta})_i, \tag{4.5}$$

where G_j means the set of nodes belonging to the cellular construction side perpendicular to Oy_j axis.

To calculate l_{ijkl} and l_{ijklrs}, which characterize the effects from the initial stresses in a finite-dimensional structure, it is thus effective to use a matrix method.

5. Examples

Example 1. (working formulas for planar structure made of thin beams). The domain L_I in the case under consideration is $[0, L_I][-h_I/2, h_I/2]$ in the coordinate system $\{\boldsymbol{l}, \boldsymbol{n}\}$ linked to the Ith beam. Under hypothesis of non-deformable normal, the following relationship takes place:

$$U_l^{\alpha\beta} = v^{\alpha\beta}(l) - nw^{\alpha\beta\prime}(l), U_n^{\alpha\beta} = w^{\alpha\beta}(l), \tag{5.1}$$

where $w^{\alpha\beta}$ is the normal deflection and $v^{\alpha\beta}$ is the axial displacement (in the local coordinate linking to the beam), $U_l^{\alpha\beta}$ is displacement in the beam considered as 2-D body, l is the coordinate along the beam axis. The upper prime means derivative in the variable l.

Substituting Eq. (5.1) into Eq. (3.7) and integrating, one obtains

$$l_{ijrsab}(\sigma) = (\text{mes } Y)^{-1} \sum_{I=1}^{N} E_I A^I_{ijrsab}(\sigma), \tag{5.2}$$

where

$$\begin{aligned} A^I_{ijrsab}(\sigma) = \int_0^L & \{J_0 v^{ab\prime}(v^{rs\prime}v^{ij\prime} + w^{rs\prime}v^{ij\prime} + v^{rs\prime}w^{ij\prime} + w^{rs\prime}w^{ij\prime}) \\ & + J_2\left[w^{ab\prime\prime}(w^{rs\prime\prime}v^{ij\prime} + v^{rs\prime}w^{ij\prime\prime} + w^{rs\prime}w^{ij\prime\prime} + w^{rs\prime\prime}w^{ij\prime}) + v^{ab\prime}w^{rs\prime\prime}w^{ij\prime\prime}\right]\}dl \\ & - \int_0^L \{J_0 v^{ab\prime}\left[\gamma_l^S\left(\gamma_l^r v^{ij\prime} + \gamma_n^r w^{ij\prime}\right) + \gamma_l^j\left(\gamma_l^i v^{rs\prime} + \gamma_n^i w^{rs\prime}\right)\right] + J_2 w^{ab\prime\prime}\left[\gamma_l^s\gamma_l^r w^{ij\prime\prime} + \gamma_l^i\gamma_l^j w^{rs\prime\prime}\right]\}\mathrm{d}l \\ & + \gamma_l^i\gamma_l^j\gamma_l^r\gamma_l^s \int_0^L J_0 v^{ab\prime}\mathrm{d}l. \end{aligned} \tag{5.3}$$

Here,

$$J_k = \int_{-h/2}^{h/2} n^k\,\mathrm{d}n, \quad k = 1,2,3 \quad (J_1 = J_3 = 0,\ J_0 = h,\ J_2 = h^3/12).$$

In Eq. (5.3) integrals are over the Ith beam axis [0,L] and h means the thickness of the Ith beam (the index I is omitted here).

Note. Although strain state in a beam has a simple form, the formula (5.3) is rather complex. It is connected with the fact that the l_{ijklrs} (σ) are expressed in terms of derivatives of $\boldsymbol{N}^{\alpha\beta}$ and cannot be expressed in terms of deformations corresponding $\boldsymbol{N}^{\alpha\beta}$, see Section 2.2.

Let us consider the cellular construction which consists of a beam. It is known (Washizu, 1982; Haug et al., 1986) that

$$v = v_- + (v_+ - v_-)l/L,\ w = w_- + \phi\_l + Al^2 + Bl^3,$$

where

$$A = 2(w_+ - w_-)/L^3 - (\phi_+ + \phi_-)/L^2, \qquad A = 3(w_+ - w_-)/L^2 - (\phi_+ + \phi_-)/L.$$

Here, $(v_+,\ v_-,\ w_+,\ w_-,\ \phi_+,\ \phi_-)$ are the axial displacements, deflections, and turning angles of the beam – the generalized displacements of the beam ends (see Section 4). Substituting these equations into Eq. (5.3) and calculating the integrals (these can be calculated in the explicit form here), one can obtain an expression for $l_{ijrsab}\ (\sigma)$ through the $(v_+, v_-, w_+, w_-, \phi_+, \phi_-)$.

Example 2. (working formulas for planar structure made of rods). To obtain the formulas for the rod structure, one can use relationship (5.1) setting $w = 0$. the formulas for calculating $l_{ijrsab}(\sigma)$ can be obtained from Eq. (5.2) and these are the following:

$$l_{ijrsab}(\sigma) = \sum_{I=1}^{N} E_I h_I A^I_{ijrsab}(\sigma), \tag{5.4}$$

where

$$A^I_{ijrsab}(\sigma) = \left(v^{ab\prime} v^{rs\prime} v^{ij\prime} - v^{ab\prime}\left(\gamma^i_I \gamma^j_I v^{rs\prime} + \gamma^r_I \gamma^s_I v^{ij\prime}\right) + \gamma^i_I \gamma^j_I \gamma^r_I \gamma^s_I v^{ab\prime}\right) L_I. \tag{5.5}$$

Here, we use the fact that the integrated functions are constants. Here, L_I means the length and h_I means the thickness of the Ith rod. As above, $v = v_- + (v_+ - v_-)l/L$, then $v' = (v_+ - \mathrm{v}_-)/L$. Here, (v_+, v_-) are the axial displacements of the beam ends (see Section 4). Substituting these equations into Eq. (5.5), one can obtain expression of $A_{ijrsab}(\sigma)$ through the (v_+, v_-).

Example 3. (**X**-shaped PC). Let us consider the planar structure with **X**-shaped PC (Fig. 3). The solution of the cellular problem for the beam cellular construction shown in Fig. 3 can be obtained in an explicit form. We use the symmetry of the cellular construction and consider one beam (1/4 of the cellular construction) indicated at the Fig. 3 as L_1. To derive $\mathbf{U}^{11}$, one needs to solve the bending-tension problem for that beam: $v'' = 0$, $w'''' = 0$, subjected to the edge conditions: $v(0) = w(0) = w'(0) = 0$, $v(\sqrt{2}) = 1/\sqrt{2}$, $w(\sqrt{2}) = -1/\sqrt{2}$, $w'(\sqrt{2}) = 0$ in which w is the normal deflection and v is the axial displacement in the local coordinate linking to the beam, with the coordinate l reckoned from zero (the center of the cellular construction) and $\sqrt{2}$ the beam length. Solving the problem, one obtains

$$v = -l/2\sqrt{2}, \qquad w = -l^3/2 + 3l^2/2\sqrt{2}. \tag{5.6}$$

Applying formula (4.5), one can calculate the homogenized elastic characteristics:

$$a_{1111}(0) = E/2\sqrt{2} + 3D, \qquad a_{2211}(0) = E/2\sqrt{2} - 3D. \tag{5.7}$$

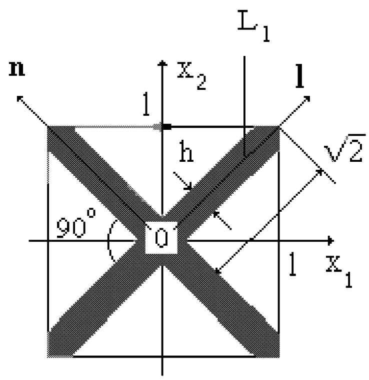

Fig. 3. X-shaped periodicity cell and the local coordinate system linking to the beam L_1.

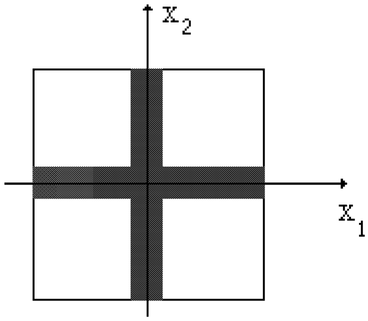

Fig. 4. Rectangular periodicity cell.

In virtue, the symmetry of PC $a_{1122} = a_{2211}$, $a_{2222} = a_{1111}$.

Some change in the method given by Kolpakov (1985) is needed to link up the elements at the faces of the PC that appear in two faces at once (Fig. 3). In the present case, by virtue of the cellular construction symmetry, the vectors for the normal faces are normal to the faces of the PC and Eq. (4.5) gives

$$a_{1111}(0) = (N + Q)/\sqrt{2} = E + 12D,$$
$$a_{1122}(0) = (N - Q)/\sqrt{2} = E - 12D. \quad (5.8)$$

($E = Eh$ and $D = Eh^3/12(1 - \nu^2)$) are the rigidities of the beam in tension and bending while N and Q are the axial and shearing forces). Due to the symmetry of PC, $a_{2222}(0) = a_{1122}(0)$, $a_{2211}(0) = a_{1122}(0)$.

Substituting Eq. (5.6) into Eqs. (5.2) and (5.3) with $ab = rs = ij = 11$, one obtains

$$l_{111111}(\sigma) = 4E \int_0^{\sqrt{2}} 1/2(J_0 + J_2)(w')^2 \mathrm{d}l = 4(J_0 + J_2)E\sqrt{2}/15.$$

Using the cellular construction symmetry one can find that $N_l^{22} = N_l^{11}$, $\mathrm{N}_n^{22} = -N_n^{11}$ and $l_{111122}(\sigma) = 4(J_0 + J_2)E\sqrt{2}/15$.

Let the homogenized (averaged) stresses be $\sigma_{11} \neq 0$, $\sigma_{mn} = 0$ for $mn \neq 11$(σ_{mn} means $\sigma_{ij}(0) = \langle \sigma_{ij}^{\varepsilon} \rangle$ as above, see Eq. (1.7)). Applying Eqs. (5.7) and (5.8), we calculate the homogenized compliance tensor: $J_{1111}(0) = (4\sqrt{2}/15)(E + 12D)/24ED$, $J_{2211}(0) = (4\sqrt{2}/15)(E + 12D)/24ED$, $J_{ij11}(0) = 0$ for ij $\neq 11$, 22.

Finally, for this case we obtain

$$a_{1111}(\sigma) = E + 12D + [(J_0 + J_2)\sqrt{2}(E + 12D)/90D + 1]\sigma_{11},$$
$$a_{1122}(\sigma) = E + 12D + [(J_0 + J_2)\sqrt{2}(E + 12D)/90D]\sigma_{11}, \quad (5.9)$$
$$a_{2222}(\sigma) = E + 12D + [(J_0 + J_2)\sqrt{2}(E + 12D)/90D]\sigma_{11} (J_0 = h, J_2 = h^3/12).$$

The last equation follows from the cellular construction symmetry.

Intermediate homogenization will give the following values for $a_{1111}(\sigma)$, $a_{1122}(\sigma)$, $a_{2222}(\sigma)$:

$$E + 12D + \sigma_{11}, E - 12D, E + 12D. \quad (5.10)$$

The discrepancies between Eqs. (5.9) and (5.10) are of the order of hE/D. The quantity hE/D is of the order h^{-1}, where h is small. Thus, this quantity can take a large value.

Example 4. (rectangular PC, Fig. 4). Consider the cellular construction shown at Fig. 4. Let the homogenized (averaged) load be applied in such way that only the $\sigma^{\varepsilon}_{11}(0)$ is not equal to zero (e.g. the weight). Solutions of cellular problem for $\alpha\beta = 11, 22$ are the following: $\boldsymbol{v}^{\alpha\beta} = y_\alpha \mathbf{e}_\beta, w^{\alpha\beta} = 0$. The local displacements coincide with the global displacements. Then, in accordance with the point 2.2 $l_{ijrs}(\sigma) = 0$.

6. Conclusions

1. The homogenization method must be applied directly to the initial structure in order to incorporate correctly the preliminary (initial) stresses if the body is non-homogeneous. Application of the direct analog of the classical theory to non-homogeneous body in general leads to an incorrect result. This conclusion concerns directly the engineering structures, which are non-homogeneous (solid-void) bodies.

2. If the initial stresses are small by comparison with the elastic constants then the first-order corrector in the homogenized constitutive equation can be computed using solution of the cellular problem for the body with no initial stresses.

3. The homogenized constitutive equation of a stressed composite body can be written in a form coinciding in form with the classical formula for stressed homogeneous body. In this representation, the term corresponding to the elastic constants depend on initial stresses. In particular, self-balanced inner stresses can affect the homogenized constants of non-homogeneous body.

4. The problem of incorporating the initial stresses represents an independent problem for every type of definite structure. In the present paper, the method of incorporating the initial stresses in application to finite-dimensional (framework and semimonocoque) constructions is given. The method proposed in the paper involves replacing the cellular problem in elasticity theory by the cellular problem in the theory of beams/plates and agrees with the analysis method for finite-dimensional structures that has been thoroughly developed in the sense of theoretical analysis and in the sense of software.

5. The working formulas are derived for stressed structures formed the basic types of structural elements: beams, rods and plates. The formulas derived here under condition that the initial stresses are small by comparison with the elastic constants are applicable for most engineering structures.

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